Axioms of an Ordered Field (`Field` & `OrderedField`)

Carrier F : Type u with primitives $0, 1 \in F$, operations $+, \cdot, -, ^{-1}$, and relations $\leq, <$. Derived operations: $a - b := a + (-b)$ and $a/b := a \cdot b^{-1}$.

I. Addition Axioms (Additive Abelian Group on F)

Addition is Associative	For all $a, b, c \in F$, $(a + b) + c = a + (b + c)$	`IsAddAssoc.add_assoc`
Addition is Commutative	For all $a, b \in F$, $a + b = b + a$	`IsAddComm.add_comm`
Left Additive Identity (0)	For all $a \in F$, $0 + a = a$	`IsZeroAdd.zero_add`
Right Additive Inverse $(-a)$	For all $a \in F$, $a + (-a) = 0$	`IsAddNeg.add_neg`
Subtraction Definition	For all $a, b \in F$, $a - b = a + (-b)$ (reverse: `add_neg_eq_sub`)	`IsSubEqAddNeg.sub_eq_add_neg`

II. Multiplication Axioms (Multiplicative Abelian Group on $F \setminus \{0\}$)

Multiplication is Associative	For all $a, b, c \in F$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$	`IsMulAssoc.mul_assoc`
Multiplication is Commutative	For all $a, b \in F$, $a \cdot b = b \cdot a$	`IsMulComm.mul_comm`
Left Multiplicative Identity (1)	For all $a \in F$, $1 \cdot a = a$	`IsOneMul.one_mul`
Right Multiplicative Inverse (a^{-1})	For all $a \in F$, $a \neq 0 \to a \cdot a^{-1} = 1$	`IsMulInvCancel.mul_inv_cancel`
Non-Triviality $(0 \neq 1)$	The additive and multiplicative identities are distinct: $(0 : F) \neq 1$	`IsZeroNeOne.zero_ne_one`
Division Definition	For all $a, b \in F$, $a/b = a \cdot b^{-1}$ (reverse: `mul_inv_eq_div`)	`IsDivEqMulInv.div_eq_mul_inv`

III. Ring Compatibility (Multiplication Distributes over Addition)

Left Distributivity	For all $a, b, c \in F$, $a \cdot (b + c) = a \cdot b + a \cdot c$	`IsMulAdd.mul_add`

IV. Non-Strict Total Order Axioms ($\leq$ is the Primary Order Engine)

Order is Reflexive	For all $a \in F$, $a \leq a$	`IsLERefl.le_refl`
Order is Transitive	For all $a, b, c \in F$, $a \leq b \to b \leq c \to a \leq c$	`IsLETrans.le_trans`
Order is Antisymmetric	For all $a, b \in F$, $a \leq b \to b \leq a \to a = b$	`IsLEAntisymm.le_antisymm`
Order is Total (Linear)	For all $a, b \in F$, $a \leq b \vee b \leq a$	`IsLETotal.le_total`
Decidable Comparison (Lean Data)	For all $a, b \in F$, `Decidable` $(a \leq b)$ (computational assumption for `if`/`max`/`min`)	`LEDecidable.decLE`

V. Strict Order Translator ($<$ Defined in Terms of $\leq$)

Strict Order Equivalence	For all $a, b \in F$, $a < b \leftrightarrow (a \leq b \wedge \neg(b \leq a))$	`IsLTIffLENotLE.lt_iff_le_not_le`

VI. Order & Field Compatibility Axioms (Linking $\leq, <$ to $+, \cdot$)

Addition Preserves $\leq$ (Right)	For all $a, b \in F$, $a \leq b \to \forall c \in F,\ a + c \leq b + c$	`IsAddLEAddRight.add_le_add_right`
Product of Positives is Positive	For all $a, b \in F$, $0 < a \to 0 < b \to 0 < a \cdot b$	`IsMulPos.mul_pos`

VII. Order Completeness and the Separate Archimedean Property

Upper Bound Definition	For $s : F \to$ `Prop` and $a \in F$, $\forall x \in F,\ s(x) \to x \leq a$	`IsUpperBound`
Least Upper Bound (sup)	`IsUpperBound` $s\ a \wedge \forall b \in F$, `IsUpperBound` $s\ b \to a \leq b$	`IsLeastUpperBound`
Order Completeness (LUB)	Every nonempty $s : F \to$ `Prop` bounded above has a least upper bound $a \in F$	`IsOrderComplete`
Archimedean Property	For all $x \in F$, $\exists n \in \mathbb{N},\ x <$ `natCast`(n) (separate mixin; follows from completeness)	`ArchimedeanProperty.archimedean`

VIII. Core Lean Structural Tools Available in Every Proof

Reflexivity & Symmetry of $=$	For all $x, y \in F$, $x = x$ and $x = y \to y = x$	`rfl` / `identity` / `h.symm`
Congruence of Functions	For all $x, y \in F$, $x = y \to f(x) = f(y)$ (e.g. add $-a$ to both sides)	`congrArg (· + -a) h`
Rewrite & Reverse Rewrite	Replace left-hand side of $=$ or $\leftrightarrow$ with right-hand side (or reverse with $\leftarrow$)	`rw [thm]` / `rw [<- thm]`

Derived Theorems & Tools in Dependency Order

Grouped by dependency layer. This is a mathematical statement reference, not a certification that every Lean proof is complete.

Group 1: Additive Group (Identities, Uniqueness, Cancellation & Subtraction)

Right & Left Additive Identity	For all $x \in F$, $x + 0 = x$ and $0 + x = x$	`add_zero` / `zero_add` [cor]
Uniqueness of Zero	For all $z \in F$, $(\forall x, z + x = x) \to z = 0$ and $(\forall x, x + z = x) \to z = 0$	`zero_unique` / `_right` [prop]
Left Additive Inverse	For all $x \in F$, $-x + x = 0$	`neg_add` [cor]
Uniqueness of Additive Inverse	For all $x, y \in F$, $x + y = 0 \to y = -x$ and $y + x = 0 \to y = -x$	`neg_unique` / `_left` [prop]
Additive Cancellation	For all $x, y, z \in F$, $x + y = x + z \to y = z$ and $y + x = z + x \to y = z$	`add_left_cancel` / `_right_` [prop]
Solve Addition via Subtraction	For all $x, a, b \in F$, $x + a = b \leftrightarrow x = b - a$ and $a + x = b \leftrightarrow x = b - a$	`add_eq_iff_eq_sub` / `_left` [prop]
Double Negation & Neg Zero	For all $x \in F$, $-(-x) = x$ and $-(0 : F) = 0$	`neg_neg` / `neg_zero` [cor]
Flip Negation Across =	For all $x, y \in F$, $-x = y \leftrightarrow x = -y$	`neg_eq_iff_eq_neg` [lem]
Self-Subtraction & Zero Test	For all $x, y \in F$, $x - x = 0$ and $x - y = 0 \leftrightarrow x = y$	`sub_self` / `sub_eq_zero_iff` [lem]
Negation of Sum & Difference	For all $x, y \in F$, $-(x + y) = -y + -x$ and $-(x - y) = y - x$	`neg_add_rev` / `neg_sub` [lem]
Add-Sub & Sub-Add Cancel	For all $x, y \in F$, $(x+y)-y = x$, $(y+x)-y = x$, $(x-y)+y = x$, $y+(x-y) = x$	`add_sub_cancel_*` / `sub_add_*` [lem]
Iterated Subtraction	For all $x, y, z \in F$, $(x - y) - z = x - (y + z)$	`sub_sub` [lem]

Group 2: Multiplicative Group on $F \setminus \{0\}$ (Identities, Inverses, Cancellation & Division)

Right Multiplicative Identity	For all $x \in F$, $x \cdot 1 = x$	`mul_one` [cor]
Uniqueness of One	For all $e \in F$, $(\forall x, x \cdot e = x) \to e = 1$ and $(\forall x, e \cdot x = x) \to e = 1$	`one_unique` / `_left` [prop]
Left Multiplicative Inverse	For all $x \in F$, $x \neq 0 \to x^{-1} \cdot x = 1$	`inv_mul_cancel` [cor]
Uniqueness of Inverse	For all $x, y \in F$, $x \neq 0 \to x \cdot y = 1 \to y = x^{-1}$ and $y \cdot x = 1 \to y = x^{-1}$	`inv_unique` / `_left` [prop]
Multiplicative Cancellation	For all $x, y, z \in F$, $x \neq 0 \to x \cdot y = x \cdot z \to y = z$ and $y \cdot x = z \cdot x \to y = z$	`mul_left_cancel` / `_right_` [prop]
Mul-Div & Div-Mul Cancel	For all $x, y \in F,\ y \neq 0 \to (x \cdot y)/y = x$, $(y \cdot x)/y = x$, $(x/y) \cdot y = x$, $y \cdot (x/y) = x$	`mul_div_cancel_*` / `div_mul_*` [lem]
Solve Multiplication via Division	For all $a, x, b \in F,\ a \neq 0 \to (a \cdot x = b \leftrightarrow x = b/a)$ and $(x \cdot a = b \leftrightarrow x = b/a)$	`mul_eq_iff_eq_div` / `_right` [prop]

Group 3: Ring Interaction (Distributivity, Multiplication by Zero & Sign Laws)

Right Distributivity	For all $a, b, c \in F$, $(a + b) \cdot c = a \cdot c + b \cdot c$	`add_mul` [cor]
Multiplication by Zero	For all $a \in F$, $a \cdot 0 = 0$ and $0 \cdot a = 0$	`mul_zero` / `zero_mul` [prop]
Negation Pulls Out of Product	For all $a, b \in F$, $(-a) \cdot b = -(a \cdot b)$ and $a \cdot (-b) = -(a \cdot b)$	`neg_mul` / `mul_neg` [lem]
Distributivity over Subtraction	For all $a, b, c \in F$, $a \cdot (b - c) = a \cdot b - a \cdot c$ and $(a - b) \cdot c = a \cdot c - b \cdot c$	`mul_sub` / `sub_mul` [lem]

Group 4: Pure Order Laws ($\leq$ and $<$: Unpackers, Transitivity & Trichotomy)

Strict Order is Irreflexive	For all $a \in F$, $\neg(a < a)$	`lt_irrefl` [lem]
Strict Implies Non-Strict ($\leq$)	For all $a, b \in F$, $a < b \to a \leq b$ (**Unpacker 1:** extracts positive half of $a < b$)	`le_of_lt` [lem]
Strict Rules Out Reverse $\leq$	For all $a, b \in F$, $a < b \to \neg(b \leq a)$ (**Unpacker 2:** extracts roadblock of $a < b$)	`not_le_of_lt` [lem]
Strict Order is Asymmetric	For all $a, b \in F$, $a < b \to \neg(b < a)$	`lt_asymm` [lem]
Strict Order is Transitive	For all $a, b, c \in F$, $a < b \to b < c \to a < c$	`lt_trans` [prop]
Mixed Transitivity ($<$ and $\leq$)	For all $a, b, c \in F$, $(a < b \to b \leq c \to a < c)$ and $(a \leq b \to b < c \to a < c)$	`lt_of_lt_of_le` / `lt_of_le_of_lt` [lem]
Non-Strict Split ($<$ or $=$)	For all $a, b \in F$, $a \leq b \leftrightarrow (a < b \vee a = b)$	`le_iff_lt_or_eq` [lem]
Law of Trichotomy	For all $a, b \in F$, $a < b \vee a = b \vee b < a$	`lt_trichotomy` [thm]

Group 5: Order & Addition (Translation Invariance, Cancellation & Sign Flip)

Addition Preserves $\leq$	For all $a, b, c \in F$, $a \leq b \to a + c \leq b + c$ and $a \leq b \to c + a \leq c + b$	`add_le_add_right` / `_left` [cor]
Addition Reflects $\leq$ (Cancel)	For all $a, b, c \in F$, $a + c \leq b + c \to a \leq b$ and $c + a \leq c + b \to a \leq b$	`le_of_add_le_add_right` / `_left` [lem]
Adding Two $\leq$ Inequalities	For all $a, b, c, d \in F$, $a \leq b \to c \leq d \to a + c \leq b + d$	`add_le_add` [prop]
Addition Preserves $<$	For all $a, b, c \in F$, $a < b \to a + c < b + c$ and $a < b \to c + a < c + b$	`add_lt_add_right` / `_left` [prop]
Addition Reflects $<$ (Cancel)	For all $a, b, c \in F$, $a + c < b + c \to a < b$ and $c + a < c + b \to a < b$	`lt_of_add_lt_add_right` / `_left` [lem]
Adding Two $<$ Inequalities	For all $a, b, c, d \in F$, $a < b \to c < d \to a + c < b + d$	`add_lt_add` [prop]
Subtraction Detects Order	For all $a, b \in F$, $(0 \leq a - b \leftrightarrow b \leq a)$ and $(0 < a - b \leftrightarrow b < a)$	`sub_nonneg` / `sub_pos` [lem]
Negation Reverses Order	For all $a, b \in F$, $(-a \leq -b \leftrightarrow b \leq a)$, $(-a < -b \leftrightarrow b < a)$, $(a < 0 \to 0 < -a)$	`neg_le_neg_iff` / `neg_lt_neg_iff` [lem]

Group 6: Order, Multiplication, Maximum & Absolute Value

Product of Nonnegatives	For all $a, b \in F$, $0 \leq a \to 0 \leq b \to 0 \leq a \cdot b$	`mul_nonneg` [lem]
Positive Factor Preserves $<, \leq$	For all $a, b, c \in F$, $(a < b \to 0 < c \to ac < bc,\ ca < cb)$ and $(a \leq b \to 0 \leq c \to ac \leq bc,\ ca \leq cb)$	`mul_lt_mul_of_pos_*` / `mul_le_*` [prop]
Negative Factor Reverses $<, \leq$	For all $a, b, c \in F$, $(a < b \to c < 0 \to bc < ac,\ cb < ca)$ and $(a \leq b \to c \leq 0 \to bc \leq ac,\ cb \leq ca)$	`mul_lt_mul_of_neg_*` / `mul_le_*` [prop]
Max Commutativity & Triangle	For all $x, y \in F$, $\max(x, y) = \max(y, x)$ and $\lvert x + y \rvert \leq \lvert x \rvert + \lvert y \rvert$ where $\lvert x \rvert := \max(x, -x)$	`max_comm` [lem] / `abs_add_le` [thm]